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Uniqueness of Best Simultaneous Approximation and Strictly Interpolating Subspaces

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It is shown that for uniqueness of best simultaneous approximation for bounded sets, the notion of interpolating subspace should be replaced by a stricter one which takes care of the weak* cluster points of the extreme points of the dual ball.

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If F, G are (nonempty) subsets of a normed linear space E, F bounded, a best simultaneous approximation to F in G is $y_0 \in G$ minimizing $r(y, F) \equiv \sup_{x \in F} \|x - y\|$. The set of best simultaneous approximations to F in G is denoted $Z_G(F)$ and called also the (relative, or restricted) *Chebyshev center* for F in G, and $r_G(F) \equiv \inf_{y \in G} r(y, F)$ is called the *Chebyshev radius* of F with respect to G. Conditions for uniqueness of Chebyshev centers (i.e., $Z_G(F)$ being a singleton) were studied by Golomb [Go], Garkavi [Ga], Laurent–Tuan [LT], Rozema–Smith [RS], Lambert–Milman [LM], Amir–Ziegler [AZ] and Sahney–Singh [SS].

Necessary and sufficient conditions for a convex G to have a Chebyshev center which is at most a singleton for all F can be summarized by

THEOREM A [AZ]. *Let G be a convex subset of the normed linear space E. Then*

(a) *The following are equivalent:*

(i) *Every compact F in E has a relative Chebyshev center in G which is at most a singleton.*

(ii) *Every pair $F = \{x, y\}$ in E has a relative Chebyshev center in G which is at most a singleton.*

(iii) *E is strictly convex with respect to G (i.e., the unit sphere of E contains no nontrivial segment parallel to a segment in G).*

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(b) *Every bounded F in E has a relative Chebyshev center in G which is at most a singleton if and only if E is uniformly convex in every direction of G (i.e., iff $0 \neq x_n - z_n = \lambda_n y \in G$, $\|x_n\| = \|z_n\| = 1$ and $\|x_n + z_n\| \to 2$ implies $\lambda_n \to 0$).*

A weaker uniqueness property had been observed by Golomb and others for interpolating subspaces. Recall that an n-dimensional subspace Y of a normed space E is called an *interpolating subspace* [ADMO] if no nontrivial linear combination of n linearly independent extreme points of the dual ball $B(E^*)$ annihilates Y. This generalizes the notion of Haar subspaces in $C[a, b]$. If $y_1,..., y_n$ are linearly independent elements of a normed space E, call $G = \{\sum_{i=1}^n c_i y_i; c_i \in J_i\}$ an *RS-set* if J_i are intervals of the types (I) a singleton, (II) a nontrivial proper closed (bounded or unbounded) interval in R, or (III) the whole line, and if every subset of $\{y_1,..., y_n\}$ consisting of all y_i with J_i of type III and some y_i with J_i of type II spans an interpolating subspace.

THEOREM B. *If G is an RS-set in a normed linear space E, then for every compact $F \subset E$ with $r_E(F) < r_G(F)$, $Z_G(F)$ is a singleton.*

The particular case when all the intervals determining G are of type III, i.e., when G is a interpolating subspace is

THEOREM B_0. *If G is an interpolating subspace of the normed linear space E, then for every compact $F \subset E$ with $r_E(F) < r_G(F)$, $Z_G(F)$ is a singleton.*

Theorem B was proved in [RS] for the case $E = C[a, b]$, and Remark 2 on p. 170 suggests the generalization to the general setting. Theorems 3 and 4 in [LM] claim that Theorems B_0 and B are valid also when F is assumed only to be bounded (and not necessarily compact). This is false, as exhibited by

EXAMPLE 1. Let $E = \{x \in C[-1, 1]; x(0) = \frac{1}{2}(x(-1) + x(1))\}$ (with the sup norm), $y_0(t) = t$, $G = \operatorname{span} y_0$, $F = \{x \in E; 0 \leqslant x(t) \leqslant 1 - |t|\}$. Since $\operatorname{ext} B(E^*) = \{\pm e_t; 0 < |t| \leqslant 1\}$ (where $e_t(x) = x(t)$), G is a one-dimensional interpolating subspace. $r_E(F) = r(\frac{1}{2}, F) = \frac{1}{2} < 1 = r_G(F) = r(\alpha y_0, F)$ for $|\alpha| \leqslant 1$.

A corrected version of the Lambert–Milman theorems should take care of the w^*-accumulation points of $\operatorname{ext} B(E^*)$. Call an n – dimensional subspace $Y = \operatorname{span}\{y_1,..., y_n\}$ of E *strictly interpolating* if no nontrivial linear combination of n linearly independent functionals in the w^*-closure $\overline{\operatorname{ext}} B(E^*)$ annihilates Y.

Other equivalent statements of the strict interpolation condition are:

(i) For every linearly independent $f_1,\dots, f_n \in \overline{\operatorname{ext}}\, B(E^*)$, $\det[f_i(y_j)] \neq 0$.

(ii) For every linearly independent $f_1,\dots,f_n \in \overline{\operatorname{ext}}\, B(E^*)$, and scalars $c_1,\dots, c_n$, there is a unique $y \in Y$ with $f_i(y) = c_i$ for $i = 1,\dots, n$.

(iii) For every linearly independent $f_1,\dots,f_n \in \overline{\operatorname{ext}}\, B(E^*)$, $E^* = Y^\perp \oplus \operatorname{span}\{f_1,\dots,f_n\}$.

Interpolating subspaces of E are trivially strictly interpolating if $\operatorname{ext} B(E^*)$ is w^*-closed, e.g., in $C(T)$ spaces (T compact) or $L^1(\mu)$ spaces, or if $\operatorname{ext} B(E^*) \cup 0$ is w^*-closed, e.g., in $C_0(T)$ spaces (T locally compact).

Call $G \subset E$ a *strictly RS-set* if it is an RS-set, and the interpolation condition is replaced by strict interpolation.

THEOREM C. *If G is a strictly RS-set in the normed space E, then for every bounded $F \subset E$ with $r_E(F) < r_G(F)$, $Z_G(F)$ is a singleton.*

In particular,

THEOREM C_0. *If G is a strictly interpolating subspace of the normed space E, then for every bounded $F \subset E$ with $r_E(F) < r_G(F)$, $Z_G(F)$ is a singleton.*

Proof of Theorem C. We shall show that if $y', y'' \in G$, and $z \in E$ satisfy $r(y', F) = r(y'', F) > r(z, F) + 2\varepsilon$, for some $\varepsilon > 0$ and $y' \neq y''$, then there is $y \in G$ with $r(y, F) < r(y', F)$. If $y' = \sum_{i=1}^n c_i' y_i$ and $y'' = \sum_{i=1}^n c_i'' y_i$ let $I_0 = \{i; J_i$ of type III or $c_i' \neq c_i''\}$. Since G is a strict RS-set, $W = \{f \in \overline{\operatorname{ext}}\, B(E^*); f(y') = f(y'')\}$ spans a subspace of dimension smaller than card I_0, and there is $y_0 = \sum_{i \in I_0} t_i y_i + \sum_{i \notin I_0} c_i' y_i$ with $f(y_0) = f(z)$ for every $f \in W$. Let $V = \{f \in \overline{\operatorname{ext}}\, B(E^*); |f(y_0) - f(z)| < \varepsilon\}$. V is relatively w^*-open, hence $\overline{\operatorname{ext}}\, B(E^*) \backslash V$ is w^*-compact and, since $f(y') \neq f(y'')$ off V, $\eta \equiv \min\{|f(y') - f(y'')|; f \in \overline{\operatorname{ext}}\, B(E^*) \backslash V\} > 0$. Let $\delta \equiv \min\{|c_i' - c_i''|; i \in I_0$ and $c_i' \neq c_i''\}$. Choose $\alpha \in (0, 1)$ with $|2\alpha(t_i - (c_i' + c_i''))| < \delta$, $4\alpha(\|y_0\| + \sup_{x \in F} \|x\|) < \eta$ and let $y = ((1-\alpha)/2)(y' + y'') + \alpha y_0 = \sum_{i \in I_0}((1-\alpha)/2)(c_i' + c_i'') + \alpha t_i) y_i + \sum_{i \notin I_0} c_i' y_i$. Then $y \in G$ by the choice of α. If $\|x - y\| > q$ for some $x \in F$, then there is some $f \in \operatorname{ext} B(E^*)$ with $|f(x) - f(y)| > q$. If $f \in V$, then $q < |f(x) - f(y)| \leqslant (1-\alpha)$ $|f(x) - f((y' + y'')/2)| + \alpha(|f(x) - f(z)| + \varepsilon) \leqslant (1-\alpha)\ r(y', F) + \alpha(r(z, F) + \varepsilon) \leqslant r(y', F) - \varepsilon\alpha$. If $f \notin V$, then $|f(x) - \frac{1}{2} f(y' + y'')| \leqslant \max(|f(x) - f(y')|, |f(x) - f(y'')|) - \frac{1}{2}|f(y') - f(y'')| \leqslant r(y', F) - \frac{1}{2}\eta$; hence $q < |f(x) - f(y)| \leqslant (1-\alpha)(r(y', F) - \frac{1}{2}\eta) + \alpha(\|x\| + \|y_0\|) \leqslant r(y', F) - \frac{1}{4}\eta$. Thus $r(y, F) \leqslant r(y', F) - \min(\frac{1}{4}\eta, \varepsilon\alpha)$.

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The mistake in [LM] stems from a blunder in the proof of Theorem 2. Lambert and Milman relate to a bounded nontrivial $F \subset E$ the space $A_F \equiv E \oplus R$ with the norm $\|(x, \lambda)\| = r(x, \lambda F)$. Their Theorem 2 claims that if $(f, \mu) \in \operatorname{ext} B(A_F^*)$ then $f \in \operatorname{ext} B(E^*)$. The proof is first carried out in the case where F is a finite set, and the reduction argument is (4) $B(A_F^*) = \bigcup\{B(A_{F'}^*);\ F' \subset F,\ F' \text{ finite}\}$ which seems to follow from $B(A_F) = \bigcap\{B(A_{F'});\ F' \subset F \text{ finite}\}$. This deduction is false, e.g.,

EXAMPLE 2. Let $F = \{e_k; k = 1, 2,...\}$ be the set of unit vectors in $E = l_1$, $f = \sum_{k=1}^{\infty} e_k^* \in l_\infty$ (the constant sequence $(1, 1, 1,...)$). The A_F norm is $\|(x, \lambda)\| = \sup_k \|x - \lambda e_k\|$, so that $\|(f, 0)\| = \sup\{|\sum_{i=1}^{\infty} x_i|;\ \sup_k \|x - \lambda e_k\| \leqslant 1\} = \sup\{\sum_{i=1}^{n} |x_i|;\ \sup_k \sum_{i \neq k} |x_i| \leqslant 1\} = 1$. If $F_n = \{e_k; k = 1,..., n\}$ then, in A_{F_n}, $\|(\sum_{k=1}^{n} (1/(n-1))\, e_k, 1/(n-1)\| = 1$, while $(f, 0)(\sum_{k=1}^{n} (1/(n-1))\, e_k, 1/(n-1)) = n/(n-1)$, so that $(f, 0) \notin B(A_{F_n}^*)$.

We can see from Example 1 that Theorem 2 is false. In this case the A_F norm on the two dimensional subspace $G \oplus R$ is the l_∞^2 norm $\|(\alpha y_0, \lambda)\| = \max(|\alpha|, |\lambda|)$, and the extreme points of the dual ball are $(0, 1)$ and $(1, 0)$. But these extreme points are restrictions of functionals $(f, \mu) \in \operatorname{ext} B(A_F^*)$, while clearly no $f \in \operatorname{ext} B(E^*)$ is 0 on G.

Equation (4) in the proof of Therem 2 is valid, however, when F is compact (or even "remotal"), since in this case $\|(x, \lambda)\|_F = \|(x, \lambda)\|_{\{z\}}$, where z is a farthest point from x in λF. Therefore, Theorem B can be proved by the proof of Theorems 2–4 in [LM]. A more direct approach is also possible, e.g.,

Sketch of a Proof of Theorem B_0. Suppose $\pm y_0 \in Z_G(F)$. By a result of Laurent and Tuan (cf. [AM] for a direct proof), there is g_0 in the w^*-closure of $\{f \in \operatorname{ext} B(E^*);\ f(x \pm y_0) = r_G(F) \text{ for some } x \in F\}$ with $g_0(y_0) = \max_{y \in G} f(y)$, which in our case means $g_0 \in G^\perp$. The Caratheodory and Krein–Milman theorems provide us with $f_0,..., f_n \in \operatorname{ext} B(E^*)$, $x_0,..., x_n \in F$ and $c_0,..., c_n \geqslant 0$ such that $\sum_{i=0}^{n} c_i = 1$, $f \sum_{i=0}^{n} c_i f_i \in G^\perp$, and $f_i(x_i \pm y_0) = r_G(F) = r_G\{x_0,..., x_n\}$, hence $f_i(y_0) = 0$ for $i = 0,..., n$. If $f = 0$, then by the same characterization of Chebyshev centers, $0 \in Z_E(F)$ and $r_E(F) = r_G(F)$, contradicting our assumption. Therefore, since G is interpolating, $f_0,..., f_n$ are linearly independent. But then, for the same reason $y_0 = 0$.

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Checking the proof of Theorem C it is seen that we can weaken the strict interpolation condition, e.g., THEOREM D_0. *If Y is a subspce of the normed space E such that for every $y_0 \in Y$, $z \in E$ and $\varepsilon > 0$ there is $y_1 \in Y$ satisfying* $\inf\{f(y_1);\ f \in \operatorname{ext} B(E^*),\ |f(z - y_0)| > \varepsilon\} > 0$, *then for every bounded $F \subset E$ with $r_E(F) < r_G(F)$, $Z_G(F)$ is at most a singleton.*

EXAMPLE 3. An infinite dimensional subspace Y satisfying the conditions of Theorem D_0: $E = \{x \in C[-1, 1];\ x(0) = 0\}$, $Y = \{x \in E;\ x$ restricted to $[0, 1]$ is a polynomial of degree $\leqslant n\}$.

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A uniqueness theorem of an individual type was given in [SS]. Theorem 1 states that if K is a (not necessarily convex) compact subset of a normed linear space E and $F \subset E$ is "uniquely remotal" with respect to K (i.e., such that every $x \in K$ has a unique farthest point in F), then $Z_K(F)$ is a single point. This is false, even if we assume K, F convex, or if we change the assumption to K being uniquely remotal with respect to F.

EXAMPLE 4. (a) E any, $K = \{x, y\}$, $F = \{(x + y)/2\}$.

(b) E non-strictly convex, $K = [x, y] \subset S(E)$, $F = \{0\}$.

(c) E any, $K = F = \{x, y\}$.

(d) $E = l^2_\infty$, $K = [-e_2, e_2]$, $F = \{e_1 + e_2\}$.

In none of these examples is $Z_K(F)$ a singleton.

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